数理解析研究所講究録 1236 巻 2001 年 112-114

GEOMETRIC ASPECTS OF LARGE DEVIATIONS FOR RANDOM WALKS ON A CRYSTAL LATTICE

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The purpose of this talk is to discuss some remarkable relations among convex polyhedra showing up in various circumstances, say Gromov-Hausdorff limits of crystal lattices, homological directions of infinite paths in finite graphs, and the large deviation property (LDP) of random walks on crystal lattices.

Let us start with a simple example. Consider the square lattice $\mathbb{Z}^2$ as a metric space with the graph-distance d. Given a positive constant ϵ, we have the metric space $(\mathbb{Z}^2, \epsilon d)$ homothetic to $(\mathbb{Z}^2, d)$. We then ask what the limit $\lim_{\epsilon\downarrow 0}(\mathbb{Z}^2, \epsilon d)$ is as ϵ tends to zero? The answer is, as we may anticipate, the Euclidean 2-space $\mathbb{R}^2$ with the taxi-cab distance. In this view, it is natural to ask what happenes for a more general infinite graph with periodicity. Graphs we would like to consider are *crystal lattices* which are defined to be abelian covering graphs of finite graphs.

Theorem 1. (1)(*a special case of Gromov's result* [2]) *Let* (X, d) *be a crystal lattice with the graph-distance. There exists a normed linear space* $(L, \|\cdot\|)$ *of finite dimension such that*

$$\lim_{\epsilon\downarrow 0}(X, \epsilon d) = (L, d_1),$$

where $d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

(2) *The unit ball* $\overline{\mathcal{D}} = \{\mathbf{x} \in L \mid \|\mathbf{x}\| \leq 1\}$ *is a polyhedron.*

Let X_0 be a finite connected graph. We denote the set of all oriented edges by E_0. Let $c = (e_1, e_2, \dots)$ be an infinite path in X_0. If the limit

$$\gamma(c) = \lim_{n\to\infty} \frac{1}{n}(e_1 + \cdots + e_n)$$

exists in the 1-chain group $C_1(X_0, \mathbb{R})$, then $\gamma(c)$ is said to be the *homological direction* of c. It is easy to see that $\gamma(c)$ is a 1-cycle so that $\gamma(c) \in H_1(X_0, \mathbb{R})$. To describe the range of homological directions, define the ℓ^1-norm on $C_1(X_0, \mathbb{R})$ by

$$\|\sum_{e\in E_0^+} a_e e\|_1 = \sum_{e\in E_0^+} |a_e|,$$

where E_0^+ is an orientation of X_0.

Theorem 2. *The range of homological directions coincides with*

$$\mathcal{D}_0 = \{\alpha \in H_1(X_0, \mathbb{R}) \mid \|\alpha\|_1 \leq 1\}.$$

Note that $\mathcal{D}_0$ is a convex polyhedron in $H_1(X_0, \mathbb{R})$, symmetric around the origin.

The convex polyhedron $\mathcal{D}_0$ is related to the combinatorics of the finite graph X_0 in the following way.

Theorem 3. 1. *$\mathcal{D}_0$ is "rational" in the sense that all extreme points of $\mathcal{D}_0$ are in $H_1(X_0, \mathbb{Q})$.*
2. *$\alpha \in H_1(X_0, \mathbb{Q})$ is a vertex of $\mathcal{D}_0$ if and only if $\alpha = c/\|c\|_1$ for a circuit (simple closed path) c in X_0.*

We shall go back to crystal lattices. To be exact, a crystal lattice X is a connected infinite graph X on which a free abelian group Γ acts as an automorphism group with a finite quotient $X_0 = \Gamma\backslash X$.

A piecewise linear map Φ of X into $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^k$ ($k = \operatorname{rank}\Gamma$) is said to be a *periodic realization* if it satisfies $\Phi(\sigma x) = \Phi(x)+\sigma$. We consider a random walk on X given by a Γ-invariant transition probability p. Given a periodic realization Φ, we put $\xi_n(c) = \Phi(x_n(c))$ for an infinite path c. We thus obtain a $\Gamma \otimes \mathbb{R}$-valued process $\{\xi_n\}_{n=0}^{\infty}$.

Now comes a discussion about large deviations principle for the process $\{\xi_n\}$.

Theorem 4. *A large deviation property holds for $\{\xi_n\}$. Namely, there exists $I : \Gamma \otimes \mathbb{R} \to [0, \infty]$, (which is called entropy function) and satisfies, for $A \subset \Gamma \otimes \mathbb{R}$,*

$$\begin{aligned}
-I(\operatorname{int}A) &\leq \liminf_{n\to\infty} \frac{1}{n} \log P_x(\frac{1}{n}\xi_n \in \operatorname{int}A) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log P_x(\frac{1}{n}\xi_n \in \overline{A}) \leq -I(\overline{A}),
\end{aligned}$$

where $I(K) = \inf\{I(\mathbf{z}) \mid \mathbf{z} \in K\}$ for $K \subset \Gamma \otimes \mathbb{R}$.

To give more details, we let

$$\langle\ ,\ \rangle : (\Gamma \otimes \mathbb{R}) \times \operatorname{Hom}(\Gamma, \mathbb{R}) \to \mathbb{R}$$

be the pairing map between $\Gamma \otimes \mathbb{R}$ and its dual $(\Gamma \otimes \mathbb{R})^* = \operatorname{Hom}(\Gamma, \mathbb{R})$, and let $\rho : H_1(X_0, \mathbb{Z}) \to \Gamma$ be the surjective homomorphism coming from the covering map $X \longrightarrow X_0$.

Lemma 5. *Let $\chi \in \operatorname{Hom}(\Gamma, \mathbb{R})$.*

1. *The limit* $\lim_{n\to\infty}\frac{1}{n}\log E(e^{\langle\xi_n,\chi\rangle}) = c(\chi)$ *exists. Here* $e^{c(\chi)}$ *is the maximal positive eigenvalue of the "twisted" transition operator associated with* χ.
2. *The function* c *is real analytic, and the hessian of* c *is strictly positive definite everywhere. Thus the correspondence* $\chi \mapsto (\nabla c)(\chi)$ *is a diffeomorphism of* $\mathrm{Hom}(\Gamma,\mathbb{R})$ *onto an open subset* U *in* $\Gamma\otimes\mathbb{R}$.

By using a general recipe in the theory of large deviation (see [1]), with the *entropy* function $I : \Gamma\otimes\mathbb{R} \to [0,\infty]$ defined by

$$I(\mathbf{z}) = \sup_{\chi}(\langle \mathbf{z},\chi\rangle - c(\chi)),$$

we have the LDP for our R.W. It should be noted that the function I assumes finite values on U. We also see

Proposition 6. $\overline{U} = \rho_{\mathbb{R}}(\mathcal{D}_0)$, *and hence is independent of* p. *Moreover*

$$\overline{U} = \{\mathbf{x}\in\Gamma\otimes\mathbb{R} \mid \|\mathbf{x}\|_1 \le 1\},$$

where

$$\|\mathbf{x}\|_1 = \inf\{\|\alpha\|_1 \mid \alpha\in H_1(X_0,\mathbb{R}), \rho_{\mathbb{R}}(\alpha) = \mathbf{x}\}.$$

Therefore $\overline{U}$ *is a convex polyhedron, symmetric around the origin, and rational in the sense that the vertices of* $\overline{U}$ *are in* $\Gamma\otimes\mathbb{Q}$.

Finally, we come back to the theorem mentioned in the begining. As an application of the LDP, we have

Theorem 7.

$$\lim_{\epsilon\downarrow 0}(X,\epsilon d) = (\Gamma\otimes\mathbb{R}, d_1),$$

where $d_1(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|_1$.

REFERENCES

[1] R. S. Ellis, *Large deviations for a general class of random vectors*, Ann. Prob. **12**(1984), 1-12.

[2] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhäuser, 1999.

[3] M. Kotani and T. Sunada, *Albanese maps and off diagonal long time asymptotics for the heat kernels*, *Comm.Math.Phys.*209(2000), 633-670.

[4] M. Kotani and T. Sunada, *Standard realizations of crystal lattices via harmonic maps*, Trans. Amer. Math. Soc. **353**(2000), 1-20

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